

# A note on minimal finite quotients of mapping class groups

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*Abstract.* We prove that the minimal nontrivial finite quotient group of the mapping class group  $\mathcal{M}_g$  of a closed orientable surface of genus  $g$  is the symplectic group  $\mathrm{PSp}_{2g}(\mathbb{Z}_2)$ , for  $g = 3$  and  $4$  (this might remain true, however, for arbitrary genus  $g > 2$ ). We discuss also some results for arbitrary genus  $g$ .

## 1. Introduction

It is an interesting but in general difficult problem to classify the finite quotients (factor groups) of certain geometrically significant infinite groups. This becomes particularly attractive if the group in question is perfect (has trivial abelianization) since in this case each finite quotient projects onto a minimal quotient which is a nonabelian finite simple group, and there is the well-known list of the finite simple groups (always understood to be nonabelian in the following).

As an example, the finite quotients of the Fuchsian triangle group of type  $(2,3,7)$  (two generators of orders two and three whose product has order seven) are the so-called Hurwitz groups, the groups of orientation-preserving diffeomorphisms of maximal possible order  $84(g-1)$  of a closed orientable surface of genus  $g$ . There is a rich literature on the classification of the Hurwitz groups, and in particular on the most significant case of simple Hurwitz groups; the smallest Hurwitz group is the projective linear or linear fractional group  $\mathrm{PSL}_2(7)$  of order 168, acting on Klein's quartic of genus three.

One of the most interesting groups in topology is the mapping class group  $\mathcal{M}_g$  of a closed orientable surface  $\mathcal{F}_g$  of genus  $g$  which is the group of orientation-preserving homeomorphisms of  $\mathcal{F}_g$  modulo the subgroup of homeomorphisms isotopic to the identity; alternatively, it is the "orientation-preserving" subgroup of index two of the outer automorphism group  $\mathrm{Out}(\pi_1(\mathcal{F}))$  of the fundamental group. It is well-known that  $\mathcal{M}_g$  is a perfect group, for  $g \geq 3$  ([Po]). By abelianizing the fundamental group  $\pi_1(\mathcal{F})$  and reducing coefficients modulo a positive integer  $k$ , we get canonical projections

$$\mathcal{M}_g \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_k) \rightarrow \mathrm{PSp}_{2g}(\mathbb{Z}_k)$$

of the mapping class group  $\mathcal{M}_g$  onto the symplectic group  $\mathrm{Sp}_{2g}(\mathbb{Z})$  and the finite projective symplectic groups  $\mathrm{PSp}_{2g}(\mathbb{Z}_k)$  (see [N]); we note that, for primes  $p$  and  $g \geq 2$ ,  $\mathrm{PSp}_{2g}(\mathbb{Z}_p)$  is a simple group with the only exception of  $\mathrm{PSp}_4(\mathbb{Z}_2)$  which is isomorphic to the symmetric group  $S_6$ . The kernel of the surjection  $\mathcal{M}_g \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$  is the Torelli group  $\mathcal{T}_g$  of all mapping classes which act trivially on the first homology of the surface  $\mathcal{F}_g$ .

It is well-known that the symplectic groups  $\mathrm{Sp}_{2g}(\mathbb{Z})$  and the linear groups  $\mathrm{SL}_n(\mathbb{Z})$  are perfect, for  $g \geq 3$  resp.  $n \geq 3$ . As a consequence of the congruence subgroup property for these groups, the following holds ( $p$  denotes a prime number):

**Theorem 1.**

- i) For  $n \geq 3$ , the finite simple quotients of the linear group  $\mathrm{SL}_n(\mathbb{Z})$  are the linear groups  $\mathrm{PSL}_n(\mathbb{Z}_p)$ .
- ii) For  $g \geq 3$ , the finite simple quotients of the symplectic group  $\mathrm{Sp}_{2n}(\mathbb{Z})$  are the symplectic groups  $\mathrm{PSp}_{2n}(\mathbb{Z}_p)$ .

Theorem 1 will be proved in section 4. For the case of mapping class groups, the following is the main result of the present note.

**Theorem 2.** For  $g = 3$  and  $4$ , the minimal nontrivial finite quotient group of the mapping class group  $\mathcal{M}_g$  of genus  $g$  is the symplectic group  $\mathrm{PSp}_{2g}(\mathbb{Z}_2)$ .

We note that the order of  $\mathrm{PSp}_{2g}(\mathbb{Z}_2) = \mathrm{Sp}_{2g}(\mathbb{Z}_2)$  is  $2^{g^2}(2^2 - 1)(2^4 - 1) \dots (2^{2g} - 1)$ , so for  $g = 3$  and  $4$  the orders are 1.451.520 and 47.377.612.800; these orders grow very fast, in fact exponentially with  $g^2$ , whereas the orders of the finite subgroups of  $\mathcal{M}_g$  grow only linearly with  $g$  (bounded above by  $84(g - 1)$ , see [Z, Theorem 2.1]).

Theorem 2 raises more questions than it answers, e.g. (even for  $g = 3$  this seems to be unknown):

- which are the finite simple quotients of  $\mathcal{M}_g$ ?
- what is the minimal index of any subgroup of  $\mathcal{M}_g$ ?

Nevertheless, the proof of the Theorem appears nontrivial and interesting: considering for  $g = 3$  and  $4$  the list of the finite simple groups of order less than that of  $\mathrm{PSp}_{2g}(\mathbb{Z}_2)$  we are able to exclude all of them by considering certain finite subgroups of  $\mathcal{M}_g$  which must inject. Since such problems are of a strongly computational character, some lists of simple groups and case-by-case analysis seem unavoidable; also, since there does not seem to be much relation between mapping class groups for different genera  $g$ , it may be difficult to generalize Theorem 2 for arbitrary  $g$  (if it remains true). Concerning the case of genus three, the only simple groups, different from  $\mathrm{PSp}_6(\mathbb{Z}_2)$  and of an order smaller than the order 4.585.351.680 of  $\mathrm{PSp}_6(\mathbb{Z}_3)$ , which we cannot exclude at moment as a

quotient of  $\mathcal{M}_3$  are the groups  ${}^3D_4(2)$ ,  $M^cL$  and  $PSU_3(\mathbb{Z}_{17}) = U_3(17)$  (in the notation of [C]). For the construction of finite quotient groups of mapping class groups, see also [Sp] and [T]: most of these groups are again closely related to the symplectic groups  $PSp_{2g}(\mathbb{Z}_k)$ .

In section 3, we prove Theorem 2 for the easier case  $g = 3$ . In section 4, we discuss some results for arbitrary genus  $g$  and then deduce Theorem 2 for the case  $g = 4$ ; we prove also the following Theorem (we note that, by a result of Wiman, for  $g \geq 2$  the maximal order of a cyclic subgroup of  $\mathcal{M}_g$  is  $4g + 2$ ).

**Theorem 3.** *For  $g \geq 3$ , let  $\phi : \mathcal{M}_g \rightarrow G$  be a surjection of  $\mathcal{M}_g$  onto a finite simple group  $G$ . Then  $G$  is isomorphic to a symplectic group  $PSp_{2g}(\mathbb{Z}_p)$ , or  $G$  has an element of order  $4g + 2$ .*

## 2. Proof of Theorem 2 for $g = 3$

Let  $G$  be a finite group of orientation-preserving diffeomorphisms of a closed surface  $\mathcal{F}_g$  of genus  $g > 1$ . Then the quotient  $\mathcal{F}_g/G$  is a closed 2-orbifold: the underlying topological space is again a closed surface of some genus  $\bar{g}$ , and there are finitely many branch points of orders  $n_1, \dots, n_k$ ; we will say that the  $G$ -action is of type  $(\bar{g}; n_1, \dots, n_k)$ .

One can give the surface  $\mathcal{F}_g$  a hyperbolic or complex structure such that  $G$  acts by isometries resp. by conformal maps of the Riemann surface, by just uniformizing the quotient orbifold  $\mathcal{F}_g/G$  by a Fuchsian group of signature  $(\bar{g}; n_1, \dots, n_k)$  (see e.g. [ZVC]). Then this Fuchsian group is obtained as the group of all lifts of elements of  $G$  to the universal covering of  $\mathcal{F}_g$  (which is the hyperbolic plane), and there is a surjection of this Fuchsian group onto  $G$  whose kernel is the universal covering group of the surface. We will say in the following that the finite  $G$ -action is given by a surjection of a Fuchsian group of type or signature  $(\bar{g}; n_1, \dots, n_k)$  onto  $G$ .

By [FK, Theorem V.3.3], every conformal map of a closed Riemann surface of genus  $g > 1$  which induces the identity on the first homology is the identity. In particular, every finite group of orientation-preserving diffeomorphisms of a closed surface of genus  $g > 1$  injects into the mapping class group  $\mathcal{M}_g$  and its quotient, the symplectic group  $PSp_{2n}(\mathbb{Z})$ , and we will speak in the following of the finite group of mapping classes  $G$  of  $\mathcal{F}_g$ , of type  $(\bar{g}; n_1, \dots, n_k)$ , determined by a surjection

$$(\bar{g}; n_1, \dots, n_k) \rightarrow G$$

of a Fuchsian group of type  $(\bar{g}; n_1, \dots, n_k)$  onto  $G$ .

As an example, the Hurwitz action of the linear fractional group  $PSL_2(7)$  on the surface  $\mathcal{F}_3$  of genus three (or Klein's quartic) is determined by a surjection (unique up to conjugation in  $PGL_2(7)$ )

$$(2, 3, 7) \rightarrow PSL_2(7)$$

of the triangle group  $(0;2,3,7)=(2,3,7)$  onto the linear fractional group  $\mathrm{PSL}_2(7)$ , so this defines a subgroup  $\mathrm{PSL}_2(7)$  of the mapping class group  $\mathcal{M}_3$ .

We will consider in the following some finite subgroups of  $\mathcal{M}_3$ , represented by finite groups of diffeomorphisms of a surface of genus three, or equivalently by surjections from Fuchsian groups. For a convenient list and a classification of the finite groups acting on a surface of genus three, see [Br].

Up to conjugation,  $\mathcal{F}_3$  has three orientation-preserving involutions which are of types  $(1;2,2,2,2) = (1;2^4)$ ,  $(0;2^8)$  (a "hyperelliptic involution") and  $(2;-)$  (a free involution). For general genus  $g$ , the following is proved in [MP].

**Proposition 1.** ([MP])

- i) If  $g \geq 3$  is odd, any involution of type  $(\frac{g-1}{2}; 2, 2, 2, 2)$  normally generates  $\mathcal{M}_g$ .
- ii) If  $g \geq 4$  is even, any involution of type  $(\frac{g}{2}; 2, 2)$  normally generates  $\mathcal{M}_g$ .

In particular, the cyclic group  $\mathbb{Z}_2$  of order two of  $\mathcal{M}_3$  generated by an involution of type  $(1;2^4)$  normally generates  $\mathcal{M}_3$  and hence maps nontrivially under any nontrivial homomorphism  $\phi : \mathcal{M}_3 \rightarrow G$ . On the other hand, we note that the mapping class represented by an involution of type  $(2^8) = (0,2^8)$  lies in kernel of the canonical surjection

$$\mathcal{M}_3 \rightarrow \mathrm{PSp}_6(\mathbb{Z}) \rightarrow \mathrm{PSp}_6(\mathbb{Z}_2).$$

We consider now the Hurwitz action of  $\mathrm{PSL}_2(7)$  on  $\mathcal{F}_3$  defined by a surjection  $\pi : (2,3,7) \rightarrow \mathrm{PSL}_2(7)$  and realizing  $\mathrm{PSL}_2(7)$  as a subgroup of  $\mathcal{M}_3$ . Up to conjugation,  $\mathrm{PSL}_2(7)$  contains a unique subgroup  $\mathbb{Z}_2$ , and the preimage  $\pi^{-1}(\mathbb{Z}_2)$  in the triangle group  $(2,3,7)$  is a Fuchsian group of signature  $(1;2^4)$  (since the subgroup  $\mathbb{Z}_2$  has index four in its normalizer in  $\mathrm{PSL}_2(7)$  which is a dihedral group of order eight). Since  $\mathrm{PSL}_2(7)$  is a simple group and an involution of type  $(1;2^4)$  normally generates  $\mathcal{M}_3$ , we have:

**Lemma 1.** *Every nontrivial group homomorphism  $\phi : \mathcal{M}_3 \rightarrow G$  injects  $\mathrm{PSL}_2(7)$ .*

Remark. The preimage  $\pi^{-1}(\mathbb{Z}_3)$  of the subgroup of order three of  $\mathrm{PSL}_2(7)$  (unique up to conjugation) is a Fuchsian group of type  $(1;3,3)$ , and  $\pi^{-1}(\mathbb{Z}_7)$  is a triangle group of type  $(7,7,7)$  (the normalizer of  $\mathbb{Z}_3$  is dihedral of order 6, that of  $\mathbb{Z}_7$  is the subgroup of  $\mathrm{PSL}_2(7)$  represented by all upper triangular matrices which has order 21). Since  $\mathrm{PSL}_2(7)$  is simple, the corresponding subgroups  $\mathbb{Z}_3$  and  $\mathbb{Z}_7$  of  $\mathcal{M}_3$  inject under any nontrivial  $\phi$ .

Now, for the proof of Theorem 2, suppose that  $\phi : \mathcal{M}_3 \rightarrow G$  is a surjection onto a nontrivial finite group  $G$  of order less than that of  $\mathrm{PSp}_6(\mathbb{Z}_2)$ ; since  $\mathcal{M}_3$  is perfect, we can assume that  $G$  is a finite nonabelian simple group. By Lemma 1,  $G$  has a subgroup

$\mathrm{PSL}_2(7)$ . The nonabelian simple groups smaller than  $\mathrm{PSp}_6(\mathbb{Z}_2)$  and having a subgroup  $\mathrm{PSL}_2(7)$  are the following (in the notation of [C, p. 239ff] to which we refer for the simple groups of small order as well as their subgroups):

$$\mathrm{L}_2(7), \mathbb{A}_7, \mathrm{U}_3(3), \mathbb{A}_8, \mathrm{L}_3(4), \mathrm{L}_2(49), \mathrm{U}_3(5), \mathbb{A}_9, \mathrm{M}_{22}, \mathrm{J}_2,$$

where  $\mathrm{L}_n(p^r) = \mathrm{PSL}_n(\mathrm{GF}(p^r))$  denotes a linear group over the Galois field with  $p^r$  elements,  $\mathrm{U}_n(p) = \mathrm{PSU}_n(\mathbb{Z}_p) = \mathrm{PSU}_n(\mathrm{GF}(p))$  a unitary and  $\mathbb{A}_n$  an alternating group;  $\mathrm{M}(22)$  is a Mathieu group and  $\mathrm{J}_2$  the second Janko or Hall-Janko group.

Now each of these groups does not have simultaneously elements of order 8, 9, or 12 (see [C]), hence the proof of Theorem 2 follows from the following:

**Lemma 2.** *There are cyclic subgroups of orders 8, 9 and 12 of  $\mathcal{M}_3$  which every nontrivial homomorphism  $\phi : \mathcal{M}_3 \rightarrow G$  injects.*

*Proof.* i) A subgroup  $\mathbb{Z}_8$  of  $\mathcal{M}_3$  is defined by a surjection  $\pi : (4, 8, 8) \rightarrow \mathbb{Z}_8$ . The preimage  $\pi^{-1}(\mathbb{Z}_2)$  is a Fuchsian group of type  $(1; 2^4)$  which hence defines a subgroup  $\mathbb{Z}_2$  of  $\mathcal{M}_3$  which normally generates  $\mathcal{M}_3$ . (See also [Sn] for the determination of the signature of a subgroup of a Fuchsian group.)

ii) A subgroup  $\mathbb{Z}_9$  of  $\mathcal{M}_3$  is defined by a surjection  $\pi : (3, 9, 9) \rightarrow \mathbb{Z}_9$ , and the preimage  $\pi^{-1}(\mathbb{Z}_3)$  gives a subgroup  $\mathbb{Z}_3$  of  $\mathcal{M}_3$  of type  $(3^5)$  (we note that, up to conjugation, there are exactly two periodic diffeomorphisms of order three of  $\mathcal{F}_3$ , of types  $(3^5)$  and  $(1; 3, 3)$ ). Suppose, by contradiction, that  $\phi$  is trivial on the subgroup  $\mathbb{Z}_3$ .

We consider a subgroup  $\mathrm{SL}_2(3)$  of  $\mathcal{M}_3$  defined by a surjection  $\pi : (3, 3, 6) \rightarrow \mathrm{SL}_2(3)$ ; the linear group  $\mathrm{SL}_2(3)$  of order 24 is isomorphic to the binary tetrahedral group  $\mathbb{A}_4^*$  and is a semidirect product  $Q_8 \rtimes \mathbb{Z}_3$ . The preimage  $\pi^{-1}(\mathbb{Z}_3)$  defines a subgroup  $\mathbb{Z}_3$  of  $\mathcal{M}_3$  of type  $(3^5)$  which, by hypothesis, is mapped trivially by  $\phi$ . Now it follows easily that  $\phi$  has to be trivial on the whole subgroup  $Q_8 \rtimes \mathbb{Z}_3$ , and in particular on the unique cyclic subgroup  $\mathbb{Z}_2$  of order two of  $Q_8$  which is of type  $(1; 2^4)$ . Since  $\mathbb{Z}_2$  normally generates  $\mathcal{M}_3$ , the homomorphism  $\phi$  is trivial.

iii) A subgroup  $\mathbb{Z}_{12}$  of  $\mathcal{M}_3$  is defined by a surjection  $\pi : (3, 4, 12) \rightarrow \mathbb{Z}_{12}$ ; now  $\pi^{-1}(\mathbb{Z}_2)$  is of type  $(1; 2^4)$ , and  $\pi^{-1}(\mathbb{Z}_3)$  of type  $(3^5)$ . Since  $\phi$  is nontrivial, it cannot be trivial on the subgroup  $\mathbb{Z}_2$  of  $\mathcal{M}_3$  of type  $(1; 2^4)$ . On the other hand, if  $\phi$  is trivial on the subgroup  $\mathbb{Z}_3$  of type  $(3^5)$  then one concludes as in ii) that  $\phi$  is trivial.

This concludes the proof of Lemma 1 and also of the case  $g = 3$  of Theorem 2.

*Remark.* The unitary group  $\mathrm{U}_3(3)$  can be excluded also by considering a quaternion subgroup  $Q_8$  of order eight of  $\mathcal{M}_3$  defined by a surjection  $\pi : (1; 2) \rightarrow Q_8$ . Again, the preimage  $\pi^{-1}(\mathbb{Z}_2)$  of the unique subgroup  $\mathbb{Z}_2$  of  $Q_8$  has signature  $(1; 2^4)$  and defines a

subgroup  $\mathbb{Z}_2$  of  $\mathcal{M}_3$ . However  $U_3(3)$  (whose Sylow 2-subgroup is a wreathed product  $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ ) has no subgroup  $Q_8$ , so  $\phi$  maps  $\mathbb{Z}_2$  trivially and hence all of  $\mathcal{M}_3$ .

### 3. Some results for arbitrary genus; proof of Theorem 2 for $g = 4$

The following is the main result of [Pa].

**Proposition 2.** ([Pa])

*For  $g \geq 3$ , the index of any proper subgroup of  $\mathcal{M}_g$  is larger than  $4g + 4$ ; equivalently, there are no surjections of  $\mathcal{M}_g$  onto an alternating group  $\mathbb{A}_n$ , or onto any transitive subgroup of  $\mathbb{A}_n$ , if  $3 \leq n \leq 4g + 4$ .*

It would be interesting to know if there exists any surjection  $\phi : \mathcal{M}_G \rightarrow \mathbb{A}_n$ , for  $n > 4g + 4$ .

The maximal order of a cyclic subgroup of  $\mathcal{M}_g$  is  $4g + 2$ , for any  $g > 1$ , and such a maximal subgroup  $\mathbb{Z}_{4g+2}$  is generated by a diffeomorphism of type  $(2, 2g + 1, 4g + 2)$ ; the subgroup  $\mathbb{Z}_2$  of  $\mathbb{Z}_{4g+2}$  is generated by a hyperelliptic involution of type  $(0; 2^{2g+2})$ . The following result is proved in [HK].

**Proposition 3.** ([HK]) *Let  $g \geq 3$ .*

- i) Let  $h$  be an orientation-preserving diffeomorphism of maximal order  $4g + 2$  of  $\mathcal{F}_g$ . If  $1 \leq k \leq 2g$  then  $h^k$  normally generates  $\mathcal{M}_g$ .*
- ii) The normal subgroup of  $\mathcal{M}_g$  generated by the hyperelliptic involution  $h^{2g+1}$  contains the Torelli group  $\mathcal{T}_g$  as a subgroup of index two and is equal to the kernel of the canonical projection  $\mathcal{M}_g \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{PSp}_{2g}(\mathbb{Z}) = \mathrm{Sp}_{2g}(\mathbb{Z})/\{\pm I\}$ .*
- iii) Let  $G$  be a group without an element of order  $g - 1$ ,  $g$  or  $2g + 1$ . Then any homomorphism  $\phi : \mathcal{M}_g \rightarrow G$  is trivial.*

Note that i) and ii) of Proposition 3 combined with Theorem 1 imply Theorem 3.

We consider the case  $g = 4$  now.

**Lemma 3.** *Let  $\phi : \mathcal{M}_4 \rightarrow G$  be a surjection onto a finite simple group  $G$ .*

- i) The symmetric group  $\mathbb{S}_5$  is a subgroup of  $G$ .*
- ii) Either  $G$  has elements of orders 10, 16 and 18, or  $G$  is isomorphic to a symplectic group  $\mathrm{PSp}_8(\mathbb{Z}_p)$ .*

*Proof.* The mapping class group  $\mathcal{M}_4$  has a subgroup  $\mathbb{A}_5$  of type  $(2, 5, 5)$  and a subgroup  $\mathbb{S}_5$  of type  $(2, 4, 5)$ . An involution in a subgroup  $\mathbb{A}_5$  of  $\mathcal{M}_4$  defines a subgroup  $\mathbb{Z}_2$  of type  $(2; 2, 2)$ ; by Proposition 1, such a subgroup  $\mathbb{Z}_2$  normally generates  $\mathcal{M}_4$ , and hence  $\phi$  injects  $\mathbb{A}_5$ ,  $\mathbb{S}_5$  and also their subgroups  $\mathbb{Z}_5$  which are of type  $(0; 5^4)$ . Now  $\mathcal{M}_4$  has a

subgroup  $\mathbb{Z}_{10}$  of type (5,10,10), and since its subgroups  $\mathbb{Z}_5$  and  $\mathbb{Z}_2$  are of type (0;5<sup>4</sup>) and (2;2,2) and hence inject, also  $\mathbb{Z}_{10}$  injects.

Also,  $\mathcal{M}_4$  has a subgroup  $\mathbb{Z}_{16}$  of type (2,16,16) whose subgroup  $\mathbb{Z}_2$  is of hyperelliptic type (0;2<sup>10</sup>), and a maximal cyclic subgroup  $\mathbb{Z}_{18}$  of type (2,9,18). Lemma 3ii) is now a consequence of Proposition 3 and Theorem 1.

*Proof of Theorem 2 for the case  $g = 4$ .*

Let  $\phi : \mathcal{M}_4 \rightarrow G$  be a surjection onto a finite simple group  $G$ . Suppose that the order of  $G$  is less than the order 47.377.612.800 of  $\mathrm{PSp}_8(\mathbb{Z}_2)$ ; see [C,p.239ff] for a list of these groups. The alternating groups of such orders are excluded by Proposition 2 since they have subgroups of index  $\leq 20$ . The linear groups  $\mathrm{PSL}_2(p^r)$  in dimension two are excluded by Lemma 3 since, with the exceptions of  $\mathrm{PSL}_2(5^2)$  and  $\mathrm{PSL}_2(5^4)$ , they have no subgroups  $\mathbb{S}_5 \cong \mathrm{PGL}_2(5)$ . All remaining groups in the list can be excluded case by case by considering the possible orders of elements in each of these groups (see [C] for the character tables of most of these groups; the group theory package GAP can also be used to create the conjugacy classes and the orders of the elements of these group). It is easy to see then that none of these groups has simultaneously elements of orders 10, 16 and 18 (in some cases it may be helpful also to consider a Sylow 2-subgroup of  $G$ ). Applying Lemma 3 again completes now the proof that the smallest (simple) quotient group of  $\mathcal{M}_4$  is indeed the symplectic group  $\mathrm{PSp}_8(\mathbb{Z}_2)$ .

#### 4. Proof of Theorem 1

i) Let  $\phi : \mathrm{SL}_n(\mathbb{Z}) \rightarrow G$  be a surjection onto a finite simple group  $G$ . By the congruence subgroup property for linear groups in dimensions  $n > 2$  (which holds also for symplectic groups, see [M],[BMS]), the kernel of  $\phi$  contains a congruence subgroup, i.e. the kernel of a canonical projection  $\mathrm{SL}_n(\mathbb{Z}) \rightarrow \mathrm{SL}_n(\mathbb{Z}_k)$ , for some positive integer  $k$ , and hence  $\phi$  induces a surjection  $\psi : \mathrm{SL}_n(\mathbb{Z}_k) \rightarrow G$  (see [N,II.21]).

If  $k = p_1^{r_1} \dots p_s^{r_s}$  is the prime decomposition,

$$\mathrm{SL}_n(\mathbb{Z}_k) \cong \mathrm{SL}_n(\mathbb{Z}_{p_1^{r_1}}) \times \dots \times \mathrm{SL}_n(\mathbb{Z}_{p_s^{r_s}})$$

(see [N,Theorem VII.11]). Now the restriction of  $\psi : \mathrm{SL}_n(\mathbb{Z}_k) \rightarrow G$  to some factor  $\mathrm{SL}_n(\mathbb{Z}_{p_i^{r_i}})$  has to be nontrivial; since  $G$  is simple, this gives some surjection  $\psi : \mathrm{SL}_n(\mathbb{Z}_{p^r}) \rightarrow G$ .

Let  $K$  denote the kernel of the canonical surjection  $\mathrm{SL}_n(\mathbb{Z}_{p^r}) \rightarrow \mathrm{SL}_n(\mathbb{Z}_p)$ , so  $K$  consists of all matrices in  $\mathrm{SL}_n(\mathbb{Z}_{p^r})$  which are congruent to the identity matrix  $I$  when entries are taken modulo  $p$ . By performing the binomial expansion of  $(I + pA)^{p^{r-1}}$  one checks easily that  $K$  is a  $p$ -group, in particular  $K$  is solvable. Then also the kernel  $K_0$  of the canonical surjection from  $\mathrm{SL}_n(\mathbb{Z}_{p^r})$  to the central quotient  $\mathrm{PSL}_n(\mathbb{Z}_p)$  of  $\mathrm{SL}_n(\mathbb{Z}_p)$  is

solvable. Since  $G$  is simple,  $\psi$  maps  $K_0$  trivially and induces a surjection from  $\mathrm{PSL}_n(\mathbb{Z}_p)$  onto  $G$ ; since  $n > 2$ ,  $\mathrm{PSL}_n(\mathbb{Z}_p)$  is simple and this surjection is an isomorphism.

ii) By [N, Theorem VII.26],

$$\mathrm{Sp}_{2n}(\mathbb{Z}_k) \cong \mathrm{Sp}_{2n}(\mathbb{Z}_{p_1^{r_1}}) \times \dots \times \mathrm{Sp}_{2n}(\mathbb{Z}_{p_s^{r_s}}),$$

and the proof is then analogous to the first case.

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